

# ON THE SUBINVARIANCE OF UNIFORM DOMAINS IN BANACH SPACES

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**ABSTRACT.** Suppose that  $E$  and  $E'$  denote real Banach spaces with dimension at least 2, that  $D \subset E$  and  $D' \subset E'$  are domains, and that  $f : D \rightarrow D'$  is a homeomorphism. In this paper, we prove the following subinvariance property for the class of uniform domains: Suppose that  $f$  is a freely quasiconformal mapping and that  $D'$  is uniform. Then the image  $f(D_1)$  of every uniform subdomain  $D_1$  in  $D$  under  $f$  is still uniform. This result answers an open problem of Väisälä in the affirmative.

## 1. INTRODUCTION AND MAIN RESULTS

The theory of quasiconformal mappings in Banach spaces was developed by Väisälä in a series of papers [23]–[27] published in the period 1990–1992. In infinite dimensional cases the classical methods such as the extremal length and conformal invariants are no longer available. Väisälä built his theory using notions from the theory of metric spaces. The basic notions are curves and their lengths as well as special classes of domains such as uniform and John domains. In fact, given a domain  $D$  in a Banach space  $E$  it is essential to consider several metric space structures, including hyperbolic type metrics of  $D$ , at the same time. The basic metrics are the norm metric of  $E$  and the quasihyperbolic and distance ratio metrics of  $D$ .

It is a natural question to study which properties of quasiconformal mappings in the Euclidean spaces have their counterparts for freely quasiconformal mappings of Banach spaces. The term "free" in this context was coined by Väisälä and it emphasizes the dimensionfree character of the results in the case of infinite dimensional Banach spaces. Due to Väisälä's work, many results of this type are already known. In the same spirit, we will investigate a subinvariance problem.

It is well known that uniform domains are subinvariant under quasiconformal mappings in  $\mathbb{R}^n$ . By this, we mean that if  $f : D \rightarrow D'$  is a  $K$ -quasiconformal mapping between domains in  $\mathbb{R}^n$  and the image domain  $D'$  is  $c$ -uniform, then  $D'_1 = f(D_1)$  is  $c'$ -uniform, for every  $c$ -uniform subdomain  $D_1 \subset D$ , where  $c' = c'(c, K, n)$ . This follows from the corresponding result for QED domains [6, Remark] and from [21, Theorem 5.6].

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Our work is motivated by these ideas which we will extend to the context of freely quasiconformal mappings in Banach spaces. In his study of the subinvariance of uniform domains in Banach spaces Väisälä proved the following invariance property under QH mappings (see Definition 5 in Section 2); see [26, Theorem 2.44].

**Theorem A.** *Suppose that  $D \subset E$  and  $D' \subset E'$ , where  $E$  and  $E'$  are Banach spaces with dimension at least 2, that the domain  $D'$  and  $G \subset D$  are  $c$ -uniform, and that  $f : D \rightarrow D'$  is  $M$ -QH. Then  $f(G)$  is  $c'$ -uniform with  $c' = c'(c, M)$ .*

A natural problem is whether the corresponding result holds for freely quasiconformal mappings (see Definition 4 in Section 2) in Banach spaces or not. In fact, the problem was raised by Väisälä in the following way; see [26, Subsection 2.43].

**Open Problem 1.** Suppose that  $D' \subset E'$  is  $a$ -uniform and that  $f : D \rightarrow D'$  is a  $\psi$ -FQC mapping (i.e., a freely  $\varphi$ -quasiconformal mapping), where  $D \subset E$ . If  $D_1$  is a  $c$ -uniform subdomain in  $D$ , is it true that then  $D'_1 = f(D_1)$  is  $c'$ -uniform with  $c' = c'(a, c, \psi)$ ?

Our main result is the following theorem, which shows that the answer to Open Problem 1 is in the affirmative.

**Theorem 1.** *Suppose that  $D' \subset E'$  is  $a$ -uniform and that  $f : D \rightarrow D'$  is a  $\psi$ -FQC mapping, where  $D \subset E$ . For each subdomain  $D_1$  in  $D$ , if  $D_1$  is  $c$ -uniform, then  $f(D_1)$  is still  $c'$ -uniform, where  $c' = c'(a, c, \psi)$ .*

We remark that the hypothesis “ $D'$  being uniform” in Theorem 1 is necessary. This can be seen by letting  $D = \mathbb{B}$  and  $D' = \mathbb{B} \setminus [0, 1)$  in  $\mathbb{R}^2$ .

The proof of Theorem 1 will be given in Section 3. In Section 2, some necessary preliminaries will be introduced.

## 2. PRELIMINARIES

**2.1. Notation.** Throughout the paper, we always assume that  $E$  and  $E'$  denote real Banach spaces with dimension at least 2. The norm of a vector  $z$  in  $E$  is written as  $|z|$ , and for every pair of points  $z_1, z_2$  in  $E$ , the distance between them is denoted by  $|z_1 - z_2|$ , the closed line segment with endpoints  $z_1$  and  $z_2$  by  $[z_1, z_2]$ . Moreover, we use  $\mathbb{B}(x, r)$  to denote the ball with center  $x \in E$  and radius  $r (\geq 0)$ , and its boundary and closure are denoted by  $\mathbb{S}(x, r)$  and  $\overline{\mathbb{B}}(x, r)$ , respectively. In particular, we use  $\mathbb{B}$  to denote the unit ball  $\mathbb{B}(0, 1)$ .

For the convenience of notation, given domains  $D \subset E$ ,  $D' \subset E'$  and a mapping  $f : D \rightarrow D'$  and points  $x, y, z, \dots$  in  $D$ , we always denote by  $x', y', z', \dots$  the images in  $D'$  of  $x, y, z, \dots$  under  $f$ , respectively. Also we assume that  $\alpha, \beta, \gamma, \dots$  denote curves in  $D$  and  $\alpha', \beta', \gamma', \dots$  the images in  $D'$  of  $\alpha, \beta, \gamma, \dots$  under  $f$ , respectively.

**2.2. Uniform domains.** In their paper [17] Martio and Sarvas introduced uniform domains. Now there are many alternative characterizations for uniform domains, see [5, 8, 16, 24, 26, 27]. The importance of this class of domains in the function theory is well documented; see [5, 21] etc. Moreover, uniform domains in  $\mathbb{R}^n$  enjoy numerous geometric and function theoretic features in many areas of modern mathematical

analysis; see [1, 3, 8, 9, 11, 12, 15, 21]. We adopt the definition of uniform domains following closely the notation and terminology of [19, 21, 22, 23, 24] or [16].

**Definition 1.** A domain  $D$  in  $E$  is called *c-uniform* in the norm metric provided there exists a constant  $c$  with the property that each pair of points  $z_1, z_2$  in  $D$  can be joined by a rectifiable arc  $\alpha$  in  $D$  satisfying

- (1)  $\min_{j=1,2} \ell(\alpha[z_j, z]) \leq c d_D(z)$  for all  $z \in \alpha$ , and
- (2)  $\ell(\alpha) \leq c |z_1 - z_2|$ ,

where  $\ell(\alpha)$  denotes the length of  $\alpha$ ,  $\alpha[z_j, z]$  the part of  $\alpha$  between  $z_j$  and  $z$ , and  $d_D(z)$  the distance from  $z$  to the boundary  $\partial D$  of  $D$ . At this time,  $\alpha$  is said to be a *double c-cone arc*.

**2.3. Quasihyperbolic metric, quasihyperbolic geodesics, solid arcs and neargeodesics.** Gehring and Palka [7] introduced the quasihyperbolic metric of a domain in  $\mathbb{R}^n$ . After that, this metric has become an important tool in geometric function theory and in its generalizations to metric spaces and to Banach spaces, see [2, 3, 4, 7, 8, 10, 14, 23, 24, 27, 30] etc. Yet, some basic questions of the quasihyperbolic geometry in Banach spaces are open. For instance, only recently the convexity of quasihyperbolic balls has been studied in [13, 18, 28] in the setup of Banach spaces.

For each pair of points  $z_1, z_2$  in  $D$ , the *distance ratio metric*  $j_D(z_1, z_2)$  between  $z_1$  and  $z_2$  is defined by

$$j_D(z_1, z_2) = \log \left( 1 + \frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}} \right).$$

The *quasihyperbolic length* of a rectifiable arc or a path  $\alpha$  in the norm metric in  $D$  is the number (cf. [7, 28]):

$$\ell_{k_D}(\alpha) = \int_{\alpha} \frac{|dz|}{d_D(z)}.$$

For each pair of points  $z_1, z_2$  in  $D$ , the *quasihyperbolic distance*  $k_D(z_1, z_2)$  between  $z_1$  and  $z_2$  is defined in the usual way:

$$k_D(z_1, z_2) = \inf \ell_{k_D}(\alpha),$$

where the infimum is taken over all rectifiable arcs  $\alpha$  joining  $z_1$  to  $z_2$  in  $D$ . For all  $z_1, z_2$  in  $D$ , we have (cf. [28])

$$\begin{aligned} (2.1) \quad k_D(z_1, z_2) &\geq \inf \left\{ \log \left( 1 + \frac{\ell(\alpha)}{\min\{d_D(z_1), d_D(z_2)\}} \right) \right\} \\ &\geq j_D(z_1, z_2) \geq \left| \log \frac{d_D(z_2)}{d_D(z_1)} \right|, \end{aligned}$$

where the infimum is taken over all rectifiable curves  $\alpha$  in  $D$  connecting  $z_1$  and  $z_2$ .

In [24], Väisälä characterized uniform domains by the quasihyperbolic metric.

**Theorem B.** ([24, Theorem 6.16]) *For a domain  $D \neq E$ , the following are quantitatively equivalent:*

- (1)  $D$  is a  $c$ -uniform domain;
- (2)  $k_D(z_1, z_2) \leq c' j_D(z_1, z_2)$  for all  $z_1, z_2 \in D$ ;
- (3)  $k_D(z_1, z_2) \leq c'_1 j_D(z_1, z_2) + d$  for all  $z_1, z_2 \in D$ .

where  $c$  and  $c'$  depend on each other, and  $c$  and  $c'_1$  depend on each other.

In the case of domains in  $\mathbb{R}^n$ , the equivalence of items (1) and (3) in Theorem D is due to Gehring and Osgood [8] and the equivalence of items (2) and (3) due to Vuorinen [29, 2.50 (2)].

Recall that an arc  $\alpha$  from  $z_1$  to  $z_2$  is a *quasihyperbolic geodesic* if  $\ell_{k_D}(\alpha) = k_D(z_1, z_2)$ . Each subarc of a quasihyperbolic geodesic is obviously a quasihyperbolic geodesic. It is known that a quasihyperbolic geodesic between every pair of points in  $E$  exists if the dimension of  $E$  is finite, see [8, Lemma 1]. This is not true in arbitrary spaces (cf. [23, Example 2.9]). In order to remedy this shortage, Väisälä introduced the following concepts [24].

**Definition 2.** Let  $\alpha$  be an arc in  $E$ . The arc may be closed, open or half open. Let  $\bar{x} = (x_0, \dots, x_n)$ ,  $n \geq 1$ , be a finite sequence of successive points of  $\alpha$ . For  $h \geq 0$ , we say that  $\bar{x}$  is  $h$ -coarse if  $k_D(x_{j-1}, x_j) \geq h$  for all  $1 \leq j \leq n$ . Let  $\Phi_k(\alpha, h)$  be the family of all  $h$ -coarse sequences of  $\alpha$ . Set

$$s_k(\bar{x}) = \sum_{j=1}^n k_D(x_{j-1}, x_j)$$

and

$$\ell_{k_D}(\alpha, h) = \sup\{s_k(\bar{x}) : \bar{x} \in \Phi_k(\alpha, h)\}$$

with the agreement that  $\ell_{k_D}(\alpha, h) = 0$  if  $\Phi_k(\alpha, h) = \emptyset$ . Then the number  $\ell_{k_D}(\alpha, h)$  is the  $h$ -coarse quasihyperbolic length of  $\alpha$ .

**Definition 3.** Let  $D$  be a domain in  $E$ . An arc  $\alpha \subset D$  is  $(\nu, h)$ -solid with  $\nu \geq 1$  and  $h \geq 0$  if

$$\ell_{k_D}(\alpha[x, y], h) \leq \nu k_D(x, y)$$

for all  $x, y \in \alpha$ . A  $(\nu, 0)$ -solid arc is said to be a  $\nu$ -neargeodesic, i.e. an arc  $\alpha \subset D$  is a  $\nu$ -neargeodesic if and only if  $\ell_{k_D}(\alpha[x, y]) \leq \nu k_D(x, y)$  for all  $x, y \in \alpha$ .

Obviously, a  $\nu$ -neargeodesic is a quasihyperbolic geodesic if and only if  $\nu = 1$ .

In [24], Väisälä got the following property concerning the existence of neargeodesics in  $E$ .

**Theorem C.** ([24, Theorem 3.3]) *Let  $\{z_1, z_2\} \subset D$  and  $\nu > 1$ . Then there is a  $\nu$ -neargeodesic in  $D$  joining  $z_1$  and  $z_2$ .*

**Theorem D.** ([24, Theorem 6.22]) *Suppose that  $\gamma \subset D \neq E$  is a  $(\nu, h)$ -solid arc with endpoints  $z_1, z_2$ , and that  $D$  is a  $c$ -uniform domain. Then there is a constant  $\mu_1 = \mu_1(\nu, h, c) \geq 1$  such that*

- (1)  $\min\{\text{diam}(\gamma[z_1, z]), \text{diam}(\gamma[z_2, z])\} \leq \mu_1 d_G(z)$  for all  $z \in \gamma$ , and
- (2)  $\text{diam}(\gamma) \leq \mu_1 \max\{|z_1 - z_2|, 2(e^h - 1) \min\{d_G(z_1), d_G(z_2)\}\}$ .

## 2.4. FQC, QH and CQH mappings.

**Definition 4.** Let  $G \neq E$  and  $G' \neq E'$  be metric spaces, and let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  be a growth function, that is, a homeomorphism with  $\varphi(t) \geq t$ . We say that a homeomorphism  $f : G \rightarrow G'$  is  $\varphi$ -semisolid if

$$k_{G'}(f(x), f(y)) \leq \varphi(k_G(x, y))$$

for all  $x, y \in G$ , and  $\varphi$ -solid if both  $f$  and  $f^{-1}$  satisfy this condition.

We say that  $f$  is *fully  $\varphi$ -semisolid* (resp. *fully  $\varphi$ -solid*) if  $f$  is  $\varphi$ -semisolid (resp.  $\varphi$ -solid) on every subdomain of  $G$ . In particular, when  $G = E$ , the corresponding subdomains are taken to be proper ones. Fully  $\varphi$ -solid mappings are also called *freely  $\varphi$ -quasiconformal mappings*, or briefly  *$\varphi$ -FQC mappings*.

If  $E = \mathbb{R}^n = E'$ , then  $f$  is FQC if and only if  $f$  is quasiconformal (cf. [23]). See [20, 30] for definitions and properties of  $K$ -quasiconformal mappings, or briefly  $K$ -QC mappings.

**Definition 5.** We say that a homeomorphism  $f : D \rightarrow D'$ , where  $D \subset E$  and  $D' \subset E'$ , is  $C$ -coarsely  $M$ -quasihyperbolic, or briefly  $(M, C)$ -CQH, in the quasihyperbolic metric if it satisfies

$$\frac{k_D(x, y) - C}{M} \leq k_{D'}(f(x), f(y)) \leq M k_D(x, y) + C$$

for all  $x, y \in D$ . An  $(M, 0)$ -CQH mapping is  $M$ -bilipschitz in the quasihyperbolic metric, or briefly  $M$ -QH.

The following result easily follows from the definitions.

**Proposition 1.** *Every  $\varphi$ -FQC mapping is an  $(M, C)$ -CQH mapping, where  $M$  and  $C$  depend only on  $\varphi$ .*

**Theorem E.** ([24, Theorem 4.15]) *For domains  $D \neq E$  and  $D' \neq E'$ , suppose that  $f : D \rightarrow D'$  is  $(M, C)$ -CQH. If  $\gamma$  is a  $(\nu, h)$ -solid arc in  $D$ , then the image arc  $\gamma'$  is  $(\nu', h_1)$ -solid in  $D'$  with  $(\nu', h_1)$  depending only on  $(M, C, \nu, h)$ .*

**2.5. Quasisymmetric mappings.** Let  $X$  be a metric space and  $\dot{X} = X \cup \{\infty\}$ . By a triple in  $X$  we mean an ordered sequence  $T = (x, a, b)$  of three distinct points in  $X$ . The ratio of  $T$  is the number

$$\rho(T) = \frac{|a - x|}{|b - x|}.$$

If  $f : X \rightarrow Y$  is an injective map, the image of a triple  $T = (x, a, b)$  is the triple  $fT = (fx, fa, fb)$ .

Suppose that  $A \subset X$ . A triple  $T = (x, a, b)$  in  $X$  is said to be a triple in the pair  $(X, A)$  if  $x \in A$  or if  $\{a, b\} \subset A$ . Equivalently, both  $|a - x|$  and  $|b - x|$  are distances from a point in  $A$ .

**Definition 6.** Let  $X$  and  $Y$  be two metric spaces, and let  $\eta : [0, \infty) \rightarrow [0, \infty)$  be a homeomorphism. Suppose  $A \subset X$ . An embedding  $f : X \rightarrow Y$  is said to be  $\eta$ -quasisymmetric relative to  $A$ , or briefly  $\eta$ -QS rel  $A$ , if  $\rho(T) \leq t$  implies that  $\rho(f(T)) \leq \eta(t)$  for each triple  $T$  in  $(X, A)$  and  $t \geq 0$ .

Thus “quasisymmetry rel  $X$ ” is equivalent to ordinary “quasisymmetry”.

**Definition 7.** Let  $0 < q < 1$ , let  $\eta$  be as in Def. 6 and let  $G, G'$  be metric spaces in  $E$  and  $E'$ , respectively. A homeomorphism  $f : G \rightarrow G'$  is  $q$ -locally  $\eta$ -quasisymmetric if  $f|_{\mathbb{B}(a,qr)}$  is  $\eta$ -QS whenever  $\mathbb{B}(a,r) \subset G$ . If  $G \neq E$ , this means that  $f|_{\mathbb{B}(a,qd_G(a))}$  is  $\eta$ -QS. When  $G = E$ , obviously,  $f$  is  $\eta$ -QS.

It is known that each  $K$ -QC (i.e.,  $K$ -quasiconformal) mapping in  $\mathbb{R}^n$  is  $q$ -locally  $\eta$ -QS for every  $q < 1$  with  $\eta = \eta(K, q, n)$ , i.e.  $\eta$  depends only on the constants  $K$ ,  $q$  and  $n$  (cf. [2, 5.23]). Conversely, each  $q$ -locally  $\eta$ -QS mapping in  $\mathbb{R}^n$  is a  $K$ -QC mapping with  $K = (\eta(1))^{n-1}$  by the metric definition of quasiconformality (cf. [23, 5.6]). Further, in [23], Väisälä proved

**Theorem F.** ([23, Theorem 5.10]) *Suppose that  $D$  and  $D'$  are domains in  $E$  and  $E'$ , respectively. For a homeomorphism  $f : D \rightarrow D'$ , the following conditions are quantitatively equivalent:*

- (1)  $f$  is  $\psi$ -FQC;
- (2) both  $f$  and  $f^{-1}$  are  $q$ -locally  $\eta$ -QS;
- (3) For every  $0 < q < 1$ , there is some  $\eta(q)$  such that both  $f$  and  $f^{-1}$  are  $q$ -locally  $\eta(q)$ -QS.

**Theorem G.** ([23, Theorem 5.13]) *Suppose that  $f : E \rightarrow D' \subset E'$  is fully  $\varphi$ -semisolid. Then the following conditions are quantitatively equivalent:*

- (1)  $D' = E'$ ;
- (2)  $f$  is  $\eta$ -QS with  $\eta = \eta(\varphi)$ ;
- (3)  $f$  is  $\psi$ -FQC with  $\psi = \psi(\varphi)$ .

### 3. THE PROOF OF THEOREM 1

Before the proof of Theorem 1 we need some preparation.

**Basic assumption** In the following, we always assume that  $f : D \rightarrow D'$  is  $\psi$ -FQC and that  $D'$  is an  $a$ -uniform domain. By Proposition 1, we assume further that both  $f$  and  $f^{-1}$  are  $(M, C)$ -CQH homeomorphisms, where  $M$  and  $C$  depend only on  $\psi$ . We see from Theorem G that  $D = E$  if and only if  $D' = E'$ . Hence, to prove Theorem 1, it suffices to consider the case  $D \neq E$  and  $D' \neq E'$ . Since  $D'$  is  $a$ -uniform, Theorem B implies that there is a constant  $a'$  such that

$$(3.1) \quad k_{D'}(\xi', \zeta') \leq a' j_{D'}(\xi', \zeta')$$

for every pair of points  $\xi', \zeta' \in D'$ , where  $a' \leq 7a^3$  (cf. [26, Theorem 2.23]). Theorem F shows that both  $f$  and  $f^{-1}$  are  $\frac{3}{4}$ -locally  $\eta$ -QS with  $\eta$  depending on  $\psi$ . Let  $D_1 \subset D$

be a  $c$ -uniform domain. Then it follows from Theorem B that there is a constant  $c'$  such that

$$(3.2) \quad k_{D_1}(u, v) \leq c' j_{D_1}(u, v)$$

for every pair of points  $u, v \in D$ , where  $c' \leq 7c^3$  (cf. [26, Theorem 2.23]). For a subdomain  $D_1$  in  $D$ , since  $D'$  is  $a$ -uniform, without loss of generality, we may assume that  $D_1$  is a proper subdomain of  $D$ . For a pair of points  $x', y' \in D'_1$ , let  $\gamma'$  be a 2-neargeodesic in  $D'_1$  joining  $x'$  and  $y'$ . It follows from Theorem E that  $\gamma$  is  $(\nu', h_1)$ -solid, where  $(\nu', h_1)$  depends only on  $(M, C)$ . Moreover, we may assume that  $d_{D'}(y') \geq d_{D'}(x')$ .

We infer from Theorem D that there exists some constant  $\mu_1 = \mu_1(c, \nu', h_1)$  such that

$$(3.3) \quad \min\{\text{diam}(\gamma[x, z]), \text{diam}(\gamma[y, z])\} \leq \mu_1 d_{D_1}(z)$$

for all  $z \in \gamma$ , and

$$(3.4) \quad \text{diam}(\gamma) \leq \mu_1 \max\{|x - y|, 2(e^{h_1} - 1) \min\{d_{D_1}(x), d_{D_1}(y)\}\},$$

where the constants  $c, \nu'$  and  $h_1$  are the same as in **Basic assumption**.

Let  $z_0 \in \gamma$  be such that

$$d_{D_1}(z_0) = \sup_{p \in \gamma} d_{D_1}(p).$$

It is possible that  $z_0 = x$  or  $y$ . Then

**Lemma 1.** (1) For all  $z \in \gamma[x, z_0]$ ,

$$|x - z| \leq \rho_1 d_{D_1}(z),$$

and for each  $z \in \gamma[y, z_0]$ ,

$$|y - z| \leq \rho_1 d_{D_1}(z);$$

(2)  $\text{diam}(\gamma) \leq \rho_1 \max\{|x - y|, 2(e^{h_1} - 1) \min\{d_{D_1}(x), d_{D_1}(y)\}\},$

where  $\rho_1 = 4\mu_1^2$  and  $\mu_1$  is the same as in (3.3).

**Proof.** By (3.3) and (3.4), to prove Lemma 1, it suffices to show the first assertion in (1), i.e. for each  $z \in \gamma[x, z_0]$ ,

$$(3.5) \quad |x - z| \leq \rho_1 d_{D_1}(z).$$

For  $z \in \gamma[x, z_0]$ , we divide the proof into two cases:

$$\min\{\text{diam}(\gamma[x, z]), \text{diam}(\gamma[y, z])\} = \text{diam}(\gamma[x, z])$$

and

$$(3.6) \quad \min\{\text{diam}(\gamma[x, z]), \text{diam}(\gamma[y, z])\} = \text{diam}(\gamma[y, z]).$$

For the former case, it follows from (3.3) that (3.5) is obvious. For the latter case, we first have the following claim.

**Claim 3.1.**  $\text{diam}(\gamma[x, z]) \leq 2\mu_1 d_{D_1}(z_0)$ .

Suppose on the contrary that

$$(3.7) \quad \text{diam}(\gamma[x, z]) > 2\mu_1 d_{D_1}(z_0).$$

Obviously, there must exist some point  $w \in \gamma[x, z]$  such that

$$\text{diam}(\gamma[z, w]) = \frac{1}{2}\text{diam}(\gamma[x, z]) \text{ and } \text{diam}(\gamma[x, w]) \geq \frac{1}{2}\text{diam}(\gamma[x, z]).$$

It follows from (3.3) and (3.7) that

$$\mu_1 d_{D_1}(w) \geq \min\{\text{diam}(\gamma[x, w]), \text{diam}(\gamma[y, w])\} \geq \frac{1}{2} \text{diam}(\gamma[x, z]) > \mu_1 d_{D_1}(z_0).$$

This obvious contradiction completes the proof of Claim 3.1.

If  $\text{diam}(\gamma[y, z]) \leq \frac{1}{2}d_{D_1}(z_0)$ , then by Claim 3.1,

$$(3.8) \quad |x - z| \leq \text{diam}(\gamma[x, z]) \leq 2\mu_1 d_{D_1}(z_0) \leq 4\mu_1 d_{D_1}(z),$$

since  $d_{D_1}(z) \geq d_{D_1}(z_0) - |z_0 - z| \geq \frac{1}{2}d_{D_1}(z_0)$ .

If  $\text{diam}(\gamma[y, z]) > \frac{1}{2}d_{D_1}(z_0)$ , then we see from Claim 3.1, (3.3) and (3.6) that

$$(3.9) \quad |x - z| \leq \text{diam}(\gamma[x, z]) \leq 2\mu_1 d_{D_1}(z_0) \leq 4\mu_1 \text{diam}(\gamma[y, z]) \leq 4\mu_1^2 d_{D_1}(z),$$

which shows that (3.5) is true.

The inequalities (3.8) and (3.9) show that (3.5) also holds with the assumption of (3.6). Hence the proof of Lemma 1 is finished.  $\square$

The following result easily follows from an argument similar to the one in the proof of Lemma 1.

**Corollary 1.** (1) For each  $x_1 \in \gamma[x, z_0]$  and for all  $z \in \gamma[x_1, z_0]$ ,

$$|x_1 - z| \leq \rho_1 d_{D_1}(z);$$

(2) For each  $y_1 \in \gamma[y, z_0]$  and for all  $z \in \gamma[y_1, z_0]$ ,

$$|y_1 - z| \leq \rho_1 d_{D_1}(z),$$

where  $\rho_1$  is the same as in Lemma 1.

For the convenience of the statements and proofs of the lemmas below, we write down the related constants:

- (1)  $\vartheta = \min\{\frac{1}{2}\eta^{-1}(\frac{1}{\rho_1}), \frac{1}{2\rho_1}\},$
- (2)  $b_1 = \max\left\{\frac{1}{\psi^{-1}(\frac{1}{8})}, 1\right\}4^{8a'c'CM\rho_1\eta(\rho_1)\psi(1)},$
- (3)  $b_2 = \frac{1}{\eta^{-1}(\vartheta)}b_1^{8a'c'CM},$
- (4)  $b_3 = 2e^{5a'c'CM}b_2,$
- (5)  $b_4 = \frac{e^{\vartheta_1(\eta(13b_3^2))^{3a'M}}}{2a'c'CM\eta^{-1}(\vartheta)},$



where  $a', c', C, M, \eta$  and  $\psi$  are the same as in **Basic assumption**,  $\rho_1$  is the constant in Lemma 1 and  $\vartheta_1 = \frac{5a'b_3^4c'M^2}{\eta^{-1}(\frac{1}{b_3^3})}$ . Obviously,  $b_k > 1$  for each  $k \in \{1, 2, 3, 4\}$ .

In what follows, we prove that  $\gamma'$  is a  $b_4$ -double cone arc in  $D_1$ . (Recall that  $\gamma'$  is a 2-neargeodesic in  $D'_1$  joining  $x'$  and  $y'$  as constructed in Basic assumption.) More precisely, we will prove that  $\gamma'$  satisfies the following conditions:

$$(3.10) \quad \min\{\ell(\gamma'[x', z']), \ell(\gamma'[y', z'])\} \leq 3b_3^2 d_{D'_1}(z') \text{ for all } z' \in \gamma'[x', y']$$

and

$$(3.11) \quad \ell(\gamma') \leq b_4|x' - y'|.$$

**3.1. The proof of (3.10).** If for each  $z' \in \gamma'[x', z'_0]$ ,

$$\ell(\gamma'[x', z']) \leq b_3^2 d_{D'_1}(z'),$$

and for each  $z' \in \gamma'[y', z'_0]$ ,

$$\ell(\gamma'[y', z']) \leq b_3^2 d_{D'_1}(z'),$$

then (3.10) is obvious. Hence the remaining case we need to consider is that there is some  $p' \in \gamma'[x', z'_0]$  such that  $\ell(\gamma'[x', p']) > b_3^2 d_{D'_1}(p')$  or there is some  $q' \in \gamma'[y', z'_0]$  such that  $\ell(\gamma'[y', q']) > b_3^2 d_{D'_1}(q')$ . Obviously, we only need to consider the former case, that is, there is some  $p' \in \gamma'[x', z'_0]$  such that

$$(3.12) \quad \ell(\gamma'[x', p']) > b_3^2 d_{D'_1}(p'),$$

since the argument for the latter one is similar.

We use  $w'_0$  to denote the first point in  $\gamma'[x', z'_0]$  in the direction from  $x'$  to  $z'_0$  such that

$$(3.13) \quad \ell(\gamma'[x', w'_0]) = b_3 d_{D'_1}(w'_0).$$

Inequality (3.12) guarantees the existence of such a point  $w'_0$ .

**Lemma 2.** *For every  $z \in \gamma[w_0, z_0]$ , we have  $d_D(z) \leq b_2 d_{D_1}(z)$ .*

**Proof.** We prove this lemma by contradiction. Suppose that there exists some point  $w_1 \in \gamma[w_0, z_0]$  such that

$$(3.14) \quad d_D(w_1) > b_2 d_{D_1}(w_1).$$

Then it follows from Corollary 1 that for all  $w \in \gamma[x, w_1]$ ,

$$(3.15) \quad |w - w_1| \leq \rho_1 d_{D_1}(w_1) < \frac{\rho_1}{b_2} d_D(w_1),$$

which yields

$$k_D(w, w_1) \leq \int_{[w, w_1]} \frac{|dz|}{d_D(z)} \leq \frac{\rho_1}{b_2 - \rho_1},$$

and so

$$k_{D'}(w', w'_1) \leq \psi(k_D(w, w_1)) \leq \psi\left(\frac{\rho_1}{b_2 - \rho_1}\right) \leq \frac{1}{8},$$

since  $f$  is  $\psi$ -FQC. Hence

$$(3.16) \quad |w' - w'_1| \leq \frac{1}{7}d_{D'}(w'_1).$$

Otherwise,  $k_{D'}(w', w'_1) \geq j_{D'}(w', w'_1) > \frac{1}{8}$ .

We know from the obvious inequality  $d_{D'}(w'_0) \geq d_{D'}(w'_1) - |w'_0 - w'_1|$  that

$$d_{D'}(w'_0) \geq \frac{6}{7}d_{D'}(w'_1).$$

Then for all  $w' \in \gamma[x', w'_1]$ , we have

$$(3.17) \quad |w' - w'_0| \leq |w' - w'_1| + |w'_1 - w'_0| \leq \frac{2}{7}d_{D'}(w'_1) \leq \frac{1}{3}d_{D'}(w'_0).$$

Let  $w'_2 \in \mathbb{S}(w'_0, d_{D'_1}(w'_0)) \cap \overline{D'_1}$ , and let  $w'_3 \in \gamma[x', w'_0]$  be such that  $\ell(\gamma'[x', w'_3]) = \frac{1}{2}\ell(\gamma'[x', w'_0])$ . It follows from the assumption “ $\gamma'$  being 2-neargeodesic”, (2.1) and (3.13) that

$$(3.18) \quad \begin{aligned} k_{D'_1}(w'_0, w'_3) &\geq \frac{1}{2}\ell_{k_{D'_1}}(\gamma'[w'_3, w'_0]) \\ &\geq \frac{1}{2}\log\left(1 + \frac{\ell(\gamma'[w'_3, w'_0])}{\min\{d_{D'_1}(w'_3), d_{D'_1}(w'_0)\}}\right) \\ &\geq \frac{1}{2}\log\left(1 + \frac{b_3}{2}\right). \end{aligned}$$

Suppose that  $d_{D'_1}(w'_0) \geq \frac{3}{2}|w'_0 - w'_3|$ . Then  $k_{D'_1}(w'_0, w'_3) \leq \int_{[w'_0, w'_3]} \frac{|dz'|}{d_{D'_1}(z')} \leq 2$ , which contradicts (3.18). Hence we have proved that

$$(3.19) \quad d_{D'_1}(w'_0) < \frac{3}{2}|w'_0 - w'_3|.$$

We infer from (3.17) and (3.19) that

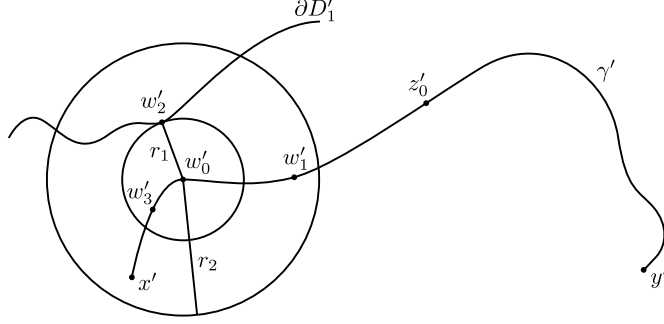
$$(3.20) \quad |w'_2 - w'_0| < \frac{3}{2}|w'_0 - w'_3| \leq \frac{1}{2}d_{D'}(w'_0).$$

Thus (3.17) and (3.20) imply that  $w'_2, w'_3 \in \mathbb{B}(w'_0, \frac{2}{3}d_{D'}(w'_0))$ . We see from Corollary 1, the choice of  $w'_2$  and the assumption “ $f^{-1}$  being  $\frac{3}{4}$ -locally  $\eta$ -QS” that

$$\frac{1}{\rho_1} \leq \frac{|w_0 - w_2|}{|w_0 - w_3|} \leq \eta\left(\frac{|w'_0 - w'_2|}{|w'_0 - w'_3|}\right).$$

Then the choice of  $w'_0$  and  $w'_3$  yields that

$$(3.21) \quad |w'_0 - w'_3| \leq \frac{1}{\eta^{-1}(\frac{1}{\rho_1})}|w'_0 - w'_2| \leq \frac{2}{\eta^{-1}(\frac{1}{\rho_1})}d_{D'_1}(w'_3).$$

FIGURE 1.  $r_1 = d_{D'_1}(w'_0), r_2 = \frac{1}{2}d_{D'}(w'_0)$ 

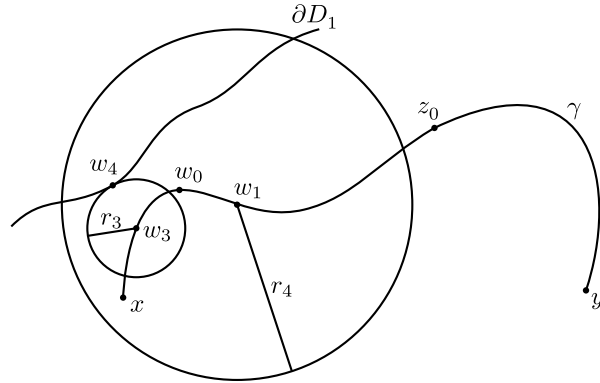
Moreover, we infer from (3.2) and (3.18) that

$$\begin{aligned} j_{D_1}(w_0, w_3) &\geq \frac{1}{c'} k_{D_1}(w_0, w_3) \geq \frac{1}{c'M} (k_{D'_1}(w'_0, w'_3) - C) \\ &> \log \left( 1 + e^{a'b_2C} \right). \end{aligned}$$

Hence

$$(3.22) \quad |w_0 - w_3| \geq e^{a'b_2C} \min\{d_{D_1}(w_3), d_{D_1}(w_0)\} > b_2 d_{D_1}(w_3),$$

since by Corollary 1,  $d_{D_1}(w_3) \leq d_{D_1}(w_0) + |w_0 - w_3| \leq (1 + \rho_1)d_{D_1}(w_0)$ .

FIGURE 2.  $r_3 = d_{D_1}(w_3), r_4 = \frac{1}{2}d_D(w_1)$ 

Let  $w_4 \in \mathbb{S}(w_3, d_{D_1}(w_3)) \cap \overline{D_1}$ . Then we obtain from (3.14) and (3.15) that

$$\begin{aligned} (3.23) \quad |w_4 - w_1| &\leq |w_4 - w_3| + |w_3 - w_1| \leq 2|w_3 - w_1| + d_{D_1}(w_1) \\ &\leq \frac{1 + 2\rho_1}{b_2} d_D(w_1), \end{aligned}$$

since  $|w_4 - w_3| = d_{D_1}(w_3) \leq d_{D_1}(w_1) + |w_3 - w_1|$ . Hence (3.15) and (3.23) show that  $w_0, w_3, w_4 \in \mathbb{B}(w_1, \frac{1}{2}d_D(w_1))$ . It follows from (3.21) and (3.22) that

$$\frac{\eta^{-1}(\frac{1}{\rho_1})}{2} \leq \frac{|w'_4 - w'_3|}{|w'_0 - w'_3|} \leq \eta\left(\frac{|w_4 - w_3|}{|w_0 - w_3|}\right) \leq \eta\left(\frac{1}{b_2}\right) < \frac{\eta^{-1}(\frac{1}{\rho_1})}{2},$$

since  $f$  is  $\frac{3}{4}$ -locally  $\eta$ -QS. This obvious contradiction completes the proof of Lemma 2.  $\square$

If  $\ell(\gamma'[y', z']) \leq b_3 d_{D'_1}(z')$  for all  $z' \in \gamma'[y', z'_0]$ , then we let  $y'_0 = z'_0$ . Otherwise, we let  $y'_0$  be the first point in  $\gamma'[y', z'_0]$  in the direction from  $y'$  to  $z'_0$  such that

$$(3.24) \quad \ell(\gamma'[y', y'_0]) = b_3 d_{D'_1}(y'_0).$$

A reasoning similar to the one in the proof of Lemma 2 shows that

**Lemma 3.** *For all  $z \in \gamma[y_0, z_0]$ , we have  $d_D(z) \leq b_2 d_{D_1}(z)$ .*

Let  $u'_0 \in \gamma'[w'_0, y'_0]$  satisfy

$$d_{D'_1}(u'_0) = \sup_{z' \in \gamma'[w'_0, y'_0]} d_{D'_1}(z').$$

Obviously, there exists a nonnegative integer  $m_1$  such that

$$2^{m_1} d_{D'_1}(w'_0) \leq d_{D'_1}(u'_0) < 2^{m_1+1} d_{D'_1}(w'_0).$$

Let  $v'_0$  be the first point in  $\gamma'[w'_0, u'_0]$  from  $w'_0$  to  $u'_0$  with

$$d_{D'_1}(v'_0) = 2^{m_1} d_{D'_1}(w'_0),$$

and let  $u'_1 = w'_0$ . If  $v'_0 = u'_1$ , we let  $u'_2 = u'_0$ . It is possible that  $u'_1 = u'_2$ . If  $v'_0 \neq u'_1$ , then we let  $u'_2, \dots, u'_{m_1+1} \in \beta'[w'_0, v'_0]$  be the points such that for each  $i \in \{2, \dots, m_1 + 1\}$ ,  $u'_i$  denotes the first point from  $w'_0$  to  $u'_0$  with

$$d_{D'_1}(u'_i) = 2^{i-1} d_{D'_1}(u'_1).$$

Obviously,  $u'_{m_1+1} = v'_0$ . If  $v'_0 \neq u'_0$ , we use  $u'_{m_1+2}$  to denote  $u'_0$ . Then we have the following assertion.

**Lemma 4.** *For all  $i \in \{1, \dots, m_1 + 1\}$  and  $z' \in \gamma'[u'_i, u'_{i+1}]$ , we have*

$$\ell(\gamma'[u'_i, u'_{i+1}]) \leq b_3 d_{D'_1}(z').$$

**Proof.** Lemma 2, Lemma 3, (3.1), (3.2) and the assumption “both  $f$  and  $f^{-1}$  being  $(M, C)$ -CQH” imply that

$$\begin{aligned}
\frac{\ell(\gamma'[u'_i, u'_{i+1}])}{2d_{D'_1}(u'_i)} &\leq \ell_{k_{D'_1}}(\gamma'[u'_i, u'_{i+1}]) \leq 2k_{D'_1}(u'_i, u'_{i+1}) \leq 2Mk_{D_1}(u_i, u_{i+1}) + 2C \\
&\leq 2c'Mj_{D_1}(u_i, u_{i+1}) + 2C \\
&\leq 2c'M \log \left( 1 + \frac{b_2|u_i - u_{i+1}|}{\min\{d_D(u_i), d_D(u_{i+1})\}} \right) + 2C \\
&\leq 2b_2c'Mk_D(u_i, u_{i+1}) + 2C \\
&\leq 2b_2c'M^2k_{D'}(u'_i, u'_{i+1}) + 2b_2c'CM + 2C \\
&\leq 2a'b_2c'M^2 \log \left( 1 + \frac{|u'_i - u'_{i+1}|}{d_{D'_1}(u'_i)} \right) + 2b_2c'CM + 2C \\
&\leq 2a'b_2c'M^2 \log \left( 1 + \frac{\ell(\gamma'[u'_i, u'_{i+1}])}{d_{D'_1}(u'_i)} \right) + 2b_2c'CM + 2C,
\end{aligned}$$

and hence we easily deduce the following conclusion:

$$(3.25) \quad \ell(\gamma'[u'_i, u'_{i+1}]) \leq (3a'b_2c'M^2)^2 d_{D'_1}(u'_i),$$

and we also see that for all  $z' \in \gamma'[u'_i, u'_{i+1}]$ ,

$$\begin{aligned}
\log \frac{d_{D'_1}(u'_i)}{d_{D'_1}(z')} &\leq k_{D'_1}(u'_i, z') \leq \ell_{k_{D'_1}}(\gamma'[u'_i, z']) \leq 2k_{D'_1}(u'_i, u'_{i+1}) \\
&\leq 2Mk_{D_1}(u_i, u_{i+1}) + 2C \\
&\leq 2c'Mj_{D_1}(u_i, u_{i+1}) + 2C \\
&\leq 2c'M \log \left( 1 + \frac{b_2|u_i - u_{i+1}|}{\min\{d_D(u_i), d_D(u_{i+1})\}} \right) + 2C \\
&\leq 2b_2c'Mk_D(u_i, u_{i+1}) + 2C \\
&\leq 2b_2c'M^2k_{D'}(u'_i, u'_{i+1}) + 2b_2c'CM + 2C \\
&\leq 2a'b_2c'M^2 \log \left( 1 + \frac{|u'_i - u'_{i+1}|}{d_{D'_1}(u'_i)} \right) + 2b_2c'CM + 2C \\
&\leq 2a'b_2c'M^2 \log \left( 1 + \frac{\ell(\gamma'[u'_i, u'_{i+1}])}{d_{D'_1}(u'_i)} \right) + 2b_2c'CM + 2C \\
&\leq 2a'b_2c'M^2 \log \left( 1 + (3a'b_2c'M^2)^2 \right) + 2b_2c'CM + 2C.
\end{aligned}$$

It follows that

$$(3.26) \quad d_{D'_1}(u'_i) \leq (4a'b_2c'M^2)^{6a'b_2c'M^2} d_{D'_1}(z').$$

Hence the proof of Lemma 4 is complete by the combination of (3.25) and (3.26).  $\square$

**Now we are ready to finish the proof of (3.10).**

For all  $z' \in \gamma'[x', u'_0]$ , if  $z' \in \gamma'[x', w'_0]$ , then (3.10) easily follows from the choice of  $w'_0$ . For the case  $z' \in \gamma'[w'_0, u'_0]$ , we know that there exists some  $k \in \{1, \dots, m_1 + 1\}$  such that  $z' \in \gamma'[u'_k, u'_{k+1}]$ . If  $k = 1$ , then by (3.13), (3.26) and Lemma 4, we know

$$\begin{aligned} \ell(\gamma'[x', z']) &= \ell(\gamma'[x', w'_0]) + \ell(\gamma'[w'_0, z']) \\ &\leq b_3(b_3 + 1)d_{D'_1}(z'). \end{aligned}$$

If  $k > 1$ , again, we obtain from (3.13), (3.26) and Lemma 4 that

$$\begin{aligned} \ell(\gamma'[x', z']) &= \ell(\gamma'[x', w'_0]) + \ell(\gamma'[w'_0, z']) \\ &\leq \ell(\gamma'[x', w'_0]) + \ell(\gamma'[u'_1, u'_2]) + \dots + \ell(\gamma'[u'_k, u'_{k+1}]) \\ &\leq b_3(2d_{D'_1}(u'_1) + d_{D'_1}(u'_2) + \dots + d_{D'_1}(u'_k)) \\ &\leq 3b_3^2 d_{D'_1}(z'). \end{aligned}$$

Hence we have proved that for all  $z' \in \gamma'[x', u'_0]$ ,

$$\ell(\gamma'[x', z']) \leq 3b_3^2 d_{D'_1}(z').$$

Similar arguments as above show that for all  $z' \in \gamma'[y', u'_0]$ ,

$$\ell(\gamma'[y', z']) \leq 3b_3^2 d_{D'_1}(z').$$

Hence the proof of (3.10) is finished.

A reasoning similar to the one in the proof of (3.10) shows that

**Corollary 2.** *For every  $x'_1 \in \gamma'[x', y']$  and all  $y'_1 \in \gamma'[x'_1, y']$ , we have that for each  $z' \in \gamma'[x'_1, y'_1]$ ,*

$$\min\{\ell(\gamma'[x'_1, z']), \ell(\gamma'[y'_1, z'])\} \leq 3b_3^2 d_{D'_1}(z').$$

**3.2. The proof of (3.11).** Suppose on the contrary that

$$(3.27) \quad \ell(\gamma'[x', y']) > b_4|x' - y'|.$$

Then we have

**Lemma 5.**  $|x' - y'| > \frac{3}{4} \max\{d_{D'_1}(x'), d_{D'_1}(y')\}.$

**Proof.** We show this lemma by contradiction. Suppose

$$|x' - y'| \leq \frac{3}{4} \max\{d_{D'_1}(x'), d_{D'_1}(y')\}.$$

Without loss of generality, we assume that

$$\max\{d_{D'_1}(x'), d_{D'_1}(y')\} = d_{D'_1}(x').$$

Then

$$d_{D'_1}(y') \geq \frac{1}{4}d_{D'_1}(x'),$$

and so

$$\begin{aligned}
(3.28) \quad \log \left( 1 + \frac{\ell(\gamma'[x', y'])}{d_{D'_1}(y')} \right) &\leq \ell_{k_{D'_1}}(\gamma'[x', y']) \leq 2k_{D'_1}(x', y') \\
&\leq 2 \int_{[x', y']} \frac{|dw'|}{d_{D'_1}(w')} \leq \frac{8|x' - y'|}{d_{D'_1}(x')} \\
&\leq 6,
\end{aligned}$$

since  $d_{D'_1}(w') \geq d_{D'_1}(x') - |x' - w'| \geq \frac{1}{4}d_{D'_1}(x')$  for all  $w' \in [x', y']$ . Thus (3.28) implies

$$\frac{\ell(\gamma'[x', y'])}{e^6 d_{D'_1}(y')} \leq \log \left( 1 + \frac{\ell(\gamma'[x', y'])}{d_{D'_1}(y')} \right) \leq \frac{8|x' - y'|}{d_{D'_1}(x')}.$$

It follows that

$$\ell(\gamma'[x', y']) \leq 8e^6|x' - y'|,$$

which contradicts (3.27). This contradiction shows that Lemma 5 holds.  $\square$

By (3.27), we see that there exist  $v'_1, v'_2 \in \gamma'[x', y']$  be such that

$$(3.29) \quad \ell(\gamma'[x', v'_1]) = 12b_3^2|x' - y'| \text{ and } \ell(\gamma'[y', v'_2]) = 12b_3^2|x' - y'|.$$

Then we have

**Lemma 6.**  $|x' - v'_1| \geq \frac{1}{2}d_{D'_1}(v'_1)$ .

**Proof.** Suppose on the contrary that

$$|x' - v'_1| < \frac{1}{2}d_{D'_1}(v'_1).$$

Then by (3.10) and (3.29), we get

$$d_{D'_1}(x') \geq d_{D'_1}(v'_1) - |x' - v'_1| \geq \frac{1}{2}d_{D'_1}(v'_1) \geq \frac{1}{6b_3^2}\ell(\gamma'[x', v'_1]) = 2|x' - y'|,$$

which contradicts Lemma 5. Hence the proof of Lemma 6 is complete.  $\square$

Similarly, we have

**Corollary 3.**  $|y' - v'_2| \geq \frac{1}{2}d_{D'_1}(v'_2)$ .

It follows from (3.10), (3.29) and Lemma 6 that

$$(3.30) \quad \min\{d_{D'_1}(v'_1), d_{D'_1}(v'_2)\} \leq d_{D'_1}(v'_1) \leq 2|x' - v'_1| \leq 24b_3^2|x' - y'|$$

and

$$(3.31) \quad |x' - v'_1| \geq \frac{1}{2}d_{D'_1}(v'_1) \geq \frac{1}{6b_3^2}\ell(\gamma'[x', v'_1]) = 2|x' - y'|,$$

whence (3.2), (3.27), (3.29) and (3.30) show that

$$\begin{aligned}
c' j_{D_1}(v_1, v_2) &\geq k_{D_1}(v_1, v_2) \geq \frac{1}{M} k_{D'_1}(v'_1, v'_2) - \frac{C}{M} \\
&\geq \frac{1}{2M} \ell_{k_{D'_1}}(\gamma'[v'_1, v'_2]) - \frac{C}{M} \\
&\geq \frac{1}{2M} \log \left( 1 + \frac{\ell(\gamma'[v'_1, v'_2])}{\min\{d_{D'_1}(v'_1), d_{D'_1}(v'_2)\}} \right) - \frac{C}{M} \\
&\geq \frac{1}{2M} \log \left( 1 + \frac{b_4 - 24b_3^2}{24b_3^2} \right) - \frac{C}{M} \\
&> \frac{1}{3M} \log \left( 1 + \frac{b_4}{b_3} \right),
\end{aligned}$$

from which we get

$$(3.32) \quad |v_1 - v_2| > \left( \frac{b_4}{b_3} \right)^{\frac{1}{3c'M}} \min\{d_{D_1}(v_1), d_{D_1}(v_2)\}.$$

Without loss of generality, we may assume that

$$\min\{d_{D_1}(v_1), d_{D_1}(v_2)\} = d_{D_1}(v_1).$$

Then we get

**Lemma 7.** *For all  $z' \in \gamma'[v'_1, v'_2]$ , we have  $d_{D'}(z') \leq b_3^3 d_{D'_1}(z')$ .*

**Proof.** Suppose on the contrary that there exists some point  $v'_3 \in \gamma'[v'_1, v'_2]$  such that

$$(3.33) \quad d_{D'}(v'_3) > b_3^3 d_{D'_1}(v'_3).$$

We divide our discussions into two cases:

$$\min\{\ell(\gamma'[x', v'_3]), \ell(\gamma'[y', v'_3])\} = \ell(\gamma'[x', v'_3])$$

and

$$\min\{\ell(\gamma'[x', v'_3]), \ell(\gamma'[y', v'_3])\} = \ell(\gamma'[y', v'_3]).$$

We only need to consider the former case since the discussion for the latter one is similar.

Obviously, Corollary 2, (3.10), (3.29) and (3.33) imply that for all  $w' \in \gamma'[x', v'_3]$ ,

$$(3.34) \quad \ell(\gamma'[w', v'_3]) \leq 3b_3^2 d_{D'_1}(v'_3) < \frac{3}{b_3} d_{D'}(v'_3),$$

and for all  $v' \in \gamma'[y', v'_2]$ ,

$$\begin{aligned}
(3.35) \quad |v'_3 - v'| &\leq |y' - v'| + |y' - x'| + |x' - v'_3| \leq \left( \frac{1}{12b_3^2} + 2 \right) \ell(\gamma'[x', v'_3]) \\
&\leq \frac{3}{b_3} \left( \frac{1}{12b_3^2} + 2 \right) d_{D'}(v'_3).
\end{aligned}$$



We know from (3.34) and (3.35) that  $x', v'_1, v'_2 \in \mathbb{B}(v'_3, \frac{1}{2}d_{D'}(v'_3))$ .

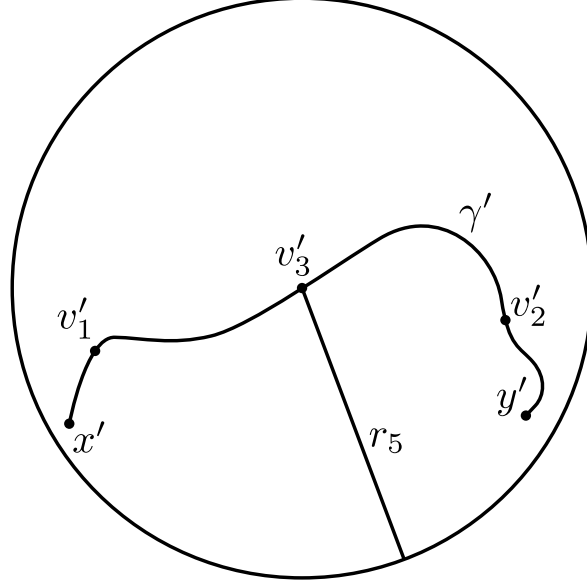


FIGURE 3.  $r_5 = \frac{1}{2}d_{D'}(v'_3)$ .

Suppose  $z_0 \in \gamma[x, v_1]$ . Then Corollary 1 implies that

$$|v_2 - v_1| \leq \rho_1 d_{D_1}(v_1),$$

which contradicts (3.32). Hence we have proved  $v_1 \in \gamma[x, z_0]$ , and then Lemma 1, (3.31) and (3.32) show that

$$(3.36) \quad \frac{1}{\rho_1} \left( \frac{b_4}{b_3} \right)^{\frac{1}{3c'M}} \leq \frac{|v_1 - v_2|}{|v_1 - x|} \leq \eta \left( \frac{|v'_1 - v'_2|}{|v'_1 - x'|} \right) < \eta(13b_3^2),$$

since  $|v'_1 - v'_2| \leq |x' - v'_1| + |y' - v'_2| + |x' - y'| \leq (24b_3^2 + 1)|x' - y'|$ . This is a contradiction, which completes the proof of Lemma 7.  $\square$

**Lemma 8.** *For all  $z \in \gamma[v_1, v_2]$ , we have  $d_D(z) \leq 2\tau d_{D_1}(z)$ , here and in the following,  $\tau = \frac{1}{\eta^{-1}(\frac{1}{b_3^3})}$ .*

**Proof.** Suppose on the contrary that there is some  $\zeta \in \gamma[v_1, v_2]$  such that

$$(3.37) \quad d_D(\zeta) > 2\tau d_{D_1}(\zeta).$$

In order to get a contradiction, we let  $z'_1 \in f(\mathbb{S}(\zeta, \frac{1}{2}d_D(\zeta))) \cap \mathbb{B}(\zeta', d_{D'}(\zeta'))$  and  $z_2 \in \mathbb{S}(\zeta, d_{D_1}(\zeta)) \cap \overline{D_1}$ . We infer from (3.37) and Lemma 7 that  $z_1, z_2 \in \mathbb{B}(\zeta, \frac{3}{4}d_D(\zeta))$ ,

$$|z'_1 - \zeta'| \geq d_{D'_1}(\zeta') \geq \frac{1}{b_3^3} d_{D'}(\zeta'),$$

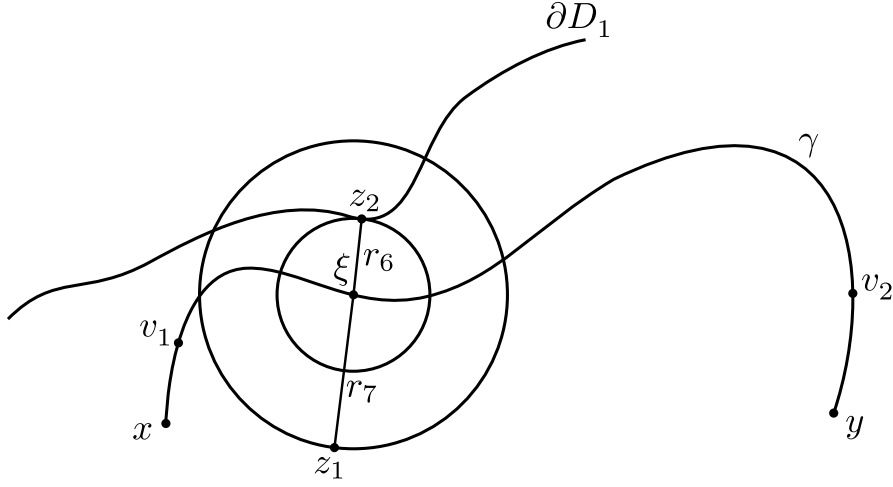


FIGURE 4.  $r_6 = d_{D_1}(\zeta), r_7 = \frac{1}{2}d_D(\zeta)$ .

$$|z'_1 - \zeta'| < d_{D'}(\zeta') \quad \text{and} \quad |z_1 - \zeta| = \frac{1}{2}d_D(\zeta) > \tau d_{D_1}(\zeta).$$

Hence

$$\frac{1}{b_3^3} \leq \frac{|z'_2 - \zeta'|}{|z'_1 - \zeta'|} \leq \eta \left( \frac{|z_2 - \zeta|}{|z_1 - \zeta|} \right) < \frac{1}{b_3^3}.$$

This obvious contradiction completes the proof of Lemma 8.  $\square$

**Now we are ready to finish the proof of (3.11).**

It follows from (3.10) and (3.29) that

$$\begin{aligned} |v'_1 - v'_2| &\leq |x' - v'_1| + |x' - y'| + |v'_2 - y'| \leq \left(2 + \frac{1}{12b_3^2}\right) \min\{\ell(\gamma'_3(x', v'_1)), \ell(\gamma'_3(y', v'_2))\} \\ &\leq \frac{1 + 24b_3^2}{4} \min\{d_{D'}(v'_1), d_{D'}(v'_2)\}, \end{aligned}$$

and then (3.1), (3.2), (3.10), (3.27), (3.29), (3.30) and Lemma 8 imply that

$$\begin{aligned}
\log \left( 1 + \frac{b_4 - 24b_3^2}{24b_3^2} \right) &\leq \log \left( 1 + \frac{\ell(\gamma'[v'_1, v'_2])}{\min\{d_{D'_1}(v'_1), d_{D'_1}(v'_2)\}} \right) \leq \ell_{k_{D'_1}}(\gamma'[v'_1, v'_2]) \\
&\leq 2k_{D'_1}(v'_1, v'_2) \leq 2Mk_{D_1}(v_1, v_2) + 2C \\
&\leq 2c'M j_{D_1}(v_1, v_2) + 2C \\
&\leq 2c'M \log \left( 1 + \frac{2\tau|v_1 - v_2|}{\min\{d_D(v_1), d_D(v_2)\}} \right) + 2C \\
&\leq 4c'\tau Mk_D(v_1, v_2) + 2C \\
&\leq 4c'\tau M^2 k_{D'}(v'_1, v'_2) + 4c'\tau CM + 2C \\
&\leq 4a'c'M^2\tau j_{D'}(v'_1, v'_2) + 4c'CM\tau + 2C \\
&\leq 4a'c'M^2\tau j_{D'_1}(v'_1, v'_2) + 4c'CM\tau + 2C \\
&< 8a'c'M^2\tau \log \frac{5 + 24b_3^2}{4}.
\end{aligned}$$

The desired contradiction shows that (3.11) is true.

**3.3. The proof of Theorem 1.** The inequalities (3.10), (3.11) and the arbitrariness of the point pair  $x'$  and  $y'$  in  $D'_1$  show that  $D'_1$  is  $c'$ -uniform, which implies that Theorem 1 is true.  $\square$

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